

Gravitational Wave Spectrums from Pole-like Inflatons based on Generalized Gravity Theories

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We present a general and unified formulation which can handle the classical evolution and quantum generation processes of the cosmological gravitational wave in a broad class of generalized gravity theories. Applications are made in several inflation models based on the scalar-tensor theory, the induced gravity, and the low energy effective action of string theories. The gravitational wave power spectrums based on the vacuum expectation value of the quantized fluctuating metric during the pole-like inflation stages are derived in analytic forms. Assuming that the gravity theory transits to Einstein one while the relevant scales remain in the superhorizon scale, we derive the consequent power spectrums and the directional fluctuations of the relic radiation produced by the gravitational wave. The spectrums seeded by the vacuum fluctuations in the pole-like inflation models based on the generalized gravity show a distinguished common feature ($n_T \simeq 3$ spectrum) which differs from the scale invariant one ($n_T \simeq 0$ spectrum) generated in an exponential inflation in Einstein gravity which is supported by observations.

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I. INTRODUCTION

Pioneering studies of the cosmological gravitational wave in Einstein gravity are made in [1–4]. The inflation generated gravitational wave spectrums are widely studied in the context of Einstein's gravity, [5–8]. Recent detection of the large angular scale fluctuations in the cosmic microwave background radiation spurred renewed interests in the potential significance of the inflation generated cosmological gravitational wave, [9,10]. For a historical review, see [10].

In a series of work we have been investigating the large scale structure formation processes in the context of early universe models based on a broad class of generalized gravity theories. We have presented the classical evolution [11] and the quantum generation [12] processes for scalar type perturbation which later evolves to the spatial density fluctuation in the large scale. Applications to scenarios with a few inflation models are made in [13]. Besides the scalar type perturbation there exist two other types: the vector and tensor types. Due to the high symmetry of the background cosmological model (the spatial homogeneity and isotropy) these three types of perturbation decouple from each other and evolve independently. The transverse vector type perturbation corresponds to the rotational mode which simply decays in an expanding medium due to the angular momentum conservation. The transverse-tracefree tensor type perturbation corresponds to the gravitational wave mode which is preserved in the superhorizon scale and is redshifted away on scales inside horizon. The equation describing the gravitational wave in the generalized gravity has a similar structure compared with the spatial curvature fluctuation of the scalar type perturbation in certain gauges; the equations and comparisons are made in [11]. Previous studies of the cosmological gravitational wave in the context of the generalized gravity can be found in [14–18,11].

In this paper we will present the general and unified formulation (both classical and quantum) for handling the cosmological gravitational wave in generalized gravity, and will apply the formulation to several inflation models based on generalized gravity theories. This paper can be considered as a gravitational wave counterpart of [13] which was concerned with the scalar type perturbation in generalized gravity. Since the gravitational field equation and the cosmological background equation will overlap we recommend to read this paper together with [13].

In Sec. II we present the classical action for the gravitational wave valid in a broad class of generalized gravity theories. We present the general asymptotic solutions and an exact solution valid under a condition which is, in fact, very general so that it includes most of the prototype inflation models known in Einstein gravity and generalized gravity. In Sec. III we present a quantum formulation and derive the normalization condition for the vacuum expectation value. In Sec. IV we apply the formulation in Sec. III to several expansion stages realized in generalized gravity theories. We derive the power spectrums based on the vacuum expectation. In Sec. V we derive the consequent classical power spectrums and the observational effects on the temperature anisotropy of the cosmic microwave background radiation. We use the conserved character of the growing mode of the gravitational wave in the large scale limit which applies independently of changes in the background equation of state and the gravity

theories. In Sec. VI comparisons are made with the power spectrums derived in the scalar type perturbation, and a brief discussion is presented.

We set $c \equiv 1$.

II. CLASSICAL EVOLUTION

We consider gravity theories included in the following action which we call a generalized $f(\phi, R)$ gravity [15]

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) \phi^{;a} \phi_{,a} - V(\phi) \right]. \quad (1)$$

The various names of generalized gravity theories which are subsets of this action are summarized in [11]. We consider a homogeneous and isotropic (flat) cosmological model with the tensor type perturbation

$$ds^2 = -dt^2 + a^2 \left(\delta_{\alpha\beta} + 2C_{\alpha\beta}^{(t)} \right) dx^\alpha dx^\beta, \quad (2)$$

where $C_{\alpha\beta}^{(t)}(\mathbf{x}, t)$ is a transverse-tracefree ($C_{\alpha}^{(t)\alpha} = 0 = C_{\alpha\beta}^{(t)|\beta}$) tensor type perturbation corresponding to the gravitational wave; the indices are based on $\delta_{\alpha\beta}$ as the metric.

The gravitational field equation and the equation of motion for the general covariant case and for the background metric are presented in Eqs. (2,5) of [13]. The second order perturbed action for the gravitational wave part of Eq. (1) becomes

$$\delta^2 S_g = \int \frac{1}{2} a^3 F \left(\dot{C}_{\beta}^{(t)\alpha} \dot{C}_{\alpha}^{(t)\beta} - \frac{1}{a^2} C_{\beta,\gamma}^{(t)\alpha} C_{\alpha}^{(t)\beta|\gamma} \right) dt d^3x, \quad (3)$$

where $F \equiv \partial f / \partial R$ and an overdot indicates a time derivative based on t . [Derivation of Eq. (3) is the following: The action expanded to second order in $C_{\alpha\beta}^{(t)}$ is presented in Eq. (18.6) of [17] in the case of Einstein gravity; see also [4,10]. Corresponding action in the case of the generalized $f(\phi, R)$ gravity can be derived by applying the conformal transformation properties presented in [19]. The result is the one in Eq. (3).] The equation of motion is

$$\ddot{C}_{\alpha\beta}^{(t)} + \left(3H + \frac{\dot{F}}{F} \right) \dot{C}_{\alpha\beta}^{(t)} - \frac{1}{a^2} \nabla^2 C_{\alpha\beta}^{(t)} = 0, \quad (4)$$

where $H \equiv \dot{a}/a$; for an equation considering the general curvature term in the background, see Eq. (102) in [16].

Equation (4) can be written as

$$\begin{aligned} v_g'' - \left(\frac{z_g''}{z_g} + \nabla^2 \right) v_g &= 0, \\ v_g(\mathbf{x}, t) &\equiv a\sqrt{F} C_{\alpha\beta}^{(t)}(\mathbf{x}, t), \quad z_g \equiv a\sqrt{F}, \end{aligned} \quad (5)$$

where a prime indicates the time derivative based on the conformal time η ($dt \equiv a d\eta$). The large and small scale asymptotic solutions can be derived, respectively, as

$$C_{\alpha\beta}^{(t)}(\mathbf{x}, t) = C_{\alpha\beta}(\mathbf{x}) - D_{\alpha\beta}(\mathbf{x}) \int_0^t \frac{1}{a^3 F} dt, \quad (6)$$

$$C_{\alpha\beta}^{(t)}(\mathbf{k}, \eta) = \frac{1}{a\sqrt{F}} \left[c_{1\alpha\beta}(\mathbf{k}) e^{ik\eta} + c_{2\alpha\beta}(\mathbf{k}) e^{-ik\eta} \right]. \quad (7)$$

Notice that in the large scale limit (larger than the visual horizon) the growing mode is conserved independently of the general changes in the background equation of state [i.e., for general $V(\phi)$] and of the changes in the gravity theories [i.e., for general $f(\phi, R)$ and $\omega(\phi)$].

For $z_g''/z_g = n_g/\eta^2$ with $n_g = \text{constant}$, Eq. (4) has an exact solution

$$C_{\alpha\beta}^{(t)}(\mathbf{k}, \eta) = \frac{\sqrt{|\eta|}}{a\sqrt{F}} \left[\tilde{c}_{1\alpha\beta}(\mathbf{k}) H_{\nu_g}^{(1)}(k|\eta|) + \tilde{c}_{2\alpha\beta}(\mathbf{k}) H_{\nu_g}^{(2)}(k|\eta|) \right], \quad \nu_g \equiv \sqrt{n_g + \frac{1}{4}}. \quad (8)$$

Using parameters

$$\epsilon_1 \equiv \frac{\dot{H}}{H^2}, \quad \epsilon_3 \equiv \frac{1}{2} \frac{\dot{F}}{HF}, \quad (9)$$

we have

$$\frac{z_g''}{z_g} = a^2 H^2 (1 + \epsilon_3) (2 + \epsilon_1 + \epsilon_3) + a^2 H \dot{\epsilon}_3. \quad (10)$$

For $\dot{\epsilon}_i = 0$ we have

$$n_g = \frac{(1 + \epsilon_3)(2 + \epsilon_1 + \epsilon_3)}{(1 + \epsilon_1)^2}, \quad (11)$$

and in this case we have the exact solution in Eq. (29). In Sec. IV we will see that most of the currently favored prototype inflation models satisfy this $n_g = \text{constant}$ condition.

We introduce a decomposition based on the two polarization states

$$\begin{aligned} C_{\alpha\beta}^{(t)}(\mathbf{x}, t) &\equiv \sqrt{\text{Vol}} \int \frac{d^3 k}{(2\pi)^3} C_{\alpha\beta}^{(t)}(\mathbf{x}, t; \mathbf{k}) \\ &\equiv \sqrt{\text{Vol}} \int \frac{d^3 k}{(2\pi)^3} \sum_{\ell} e^{i\mathbf{k} \cdot \mathbf{x}} h_{\ell\mathbf{k}}(t) e_{\alpha\beta}^{(\ell)}(\mathbf{k}), \end{aligned} \quad (12)$$

where $\ell = +, \times$; $e_{\alpha\beta}^{(+)}$ and $e_{\alpha\beta}^{(\times)}$ are bases of plus (+) and cross (×) polarization states with

$$e_{\alpha\beta}^{(\ell)}(\mathbf{k}) e^{(\ell')\alpha\beta}(\mathbf{k}) = 2\delta_{\ell\ell'}. \quad (13)$$

[In [11] we used a transverse and tracefree harmonic function $C_{\alpha\beta}^{(t)} \equiv H_T(t) Y_{\alpha\beta}^{(t)}$, see [20].] We introduce

$$\begin{aligned} h_{\ell}(\mathbf{x}, t) &\equiv \frac{1}{2} \sqrt{\text{Vol}} \int \frac{d^3 k}{(2\pi)^3} C_{\alpha\beta}^{(t)}(\mathbf{x}, t; \mathbf{k}) e^{(\ell)\alpha\beta}(\mathbf{k}) \\ &= \sqrt{\text{Vol}} \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} h_{\ell\mathbf{k}}(t). \end{aligned} \quad (14)$$

The classical power spectrum is introduced as

$$\mathcal{P}_{C_{\alpha\beta}^{(t)}}(\mathbf{k}, t) \equiv \frac{k^3}{2\pi^2} \int \langle C_{\alpha\beta}^{(t)}(\mathbf{x} + \mathbf{r}, t) C^{(t)\alpha\beta}(\mathbf{x}, t) \rangle_{\mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{r}} d^3 r, \quad (15)$$

where $\langle \rangle_{\mathbf{x}}$ is a spatial average over \mathbf{x} . We can show that

$$\mathcal{P}_{C_{\alpha\beta}^{(t)}}(\mathbf{k}, t) = 2 \sum_{\ell} \mathcal{P}_{h_{\ell}}(\mathbf{k}, t) = 2 \sum_{\ell} \frac{k^3}{2\pi^2} |h_{\ell\mathbf{k}}(t)|^2. \quad (16)$$

In a space without any preferred direction we have $h_{+\mathbf{k}} = h_{\times\mathbf{k}} = h_{\mathbf{k}}$; thus, the two polarization states contribute equally.

It is well known that the various generalized gravity theories which are subset of Eq. (1) can be transformed to Einstein gravity with a minimally coupled scalar field by conformal rescaling of the metric and field [21,15]. In [15,19] we derived the background and perturbed sets of equations valid in the generalized gravity theories by conformal rescaling of the known results in Einstein gravity. From [15,19] we find:

$$\tilde{g}_{ab} \equiv \Omega^2 g_{ab}, \quad \Omega \equiv \sqrt{F}, \quad \tilde{a} = \Omega a, \quad d\tilde{t} = \Omega dt, \quad \tilde{\eta} = \eta, \quad \tilde{C}_{\alpha\beta}^{(t)} = C_{\alpha\beta}^{(t)}, \quad \tilde{\mathbf{k}} = \mathbf{k}, \quad (17)$$

where the tilde indicates a quantity in the conformally transformed Einstein frame. Using Eq. (17) we can show that the conformal transformation of Eqs. (3-16) correctly reproduce the equations valid in Einstein gravity, [19].

In the matter dominated era with Einstein's gravity we have $F = 1/(8\pi G)$ and $a \propto t^{2/3}$. From Eq. (9) we have $\epsilon_1 = -\frac{3}{2}$ and $\epsilon_3 = 0$, thus $n_g = 2$ and $\nu_g = \frac{3}{2}$. In terms of the spherical Bessel function we have $h_{\mathbf{k}} \propto$

$j_1(k\eta)/(k\eta)$, $n_1(k\eta)/(k\eta)$. After a long period of evolution in the superhorizon scale ($k\eta \ll 1$) only the growing mode (the first term) will survive. Conventionally, we let

$$h_{\mathbf{k}}(t) \equiv A_T^{1/2}(\mathbf{k})k^{-3/2} \left[\frac{3j_1(k\eta)}{k\eta} \right]. \quad (18)$$

From Eq. (16) the power spectrum becomes

$$\mathcal{P}_{C_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, t) = \frac{\sqrt{2}}{\pi} A_T^{1/2}(\mathbf{k}) \left| \frac{3j_1(k\eta)}{k\eta} \right|. \quad (19)$$

One often writes

$$A_T(\mathbf{k}) \equiv A_T k^{n_T}. \quad (20)$$

At the horizon crossing epoch ($\frac{k}{aH}|_{\text{HC}} \equiv 1$) we have $k\eta = 2\frac{k}{aH} = 2$, and thus a *scale invariant spectrum* corresponds to $n_T = 0$.

III. QUANTUM GENERATION

The accelerated expansion stage (inflation) can generate the stochastic gravitational wave by rapidly stretching the quantum vacuum fluctuations of the perturbed metric to superhorizon scale. In order to handle the quantum mechanical generation of the gravitational wave we consider Hilbert space operator $\hat{C}_{\alpha\beta}^{(t)}$ instead of the classical metric perturbation $C_{\alpha\beta}^{(t)}$. We write (see [10])

$$\begin{aligned} \hat{C}_{\alpha\beta}^{(t)}(\mathbf{x}, t) &\equiv \int \frac{d^3k}{(2\pi)^{3/2}} \hat{C}_{\alpha\beta}^{(t)}(\mathbf{x}, t; \mathbf{k}) \\ &\equiv \int \frac{d^3k}{(2\pi)^{3/2}} \left[\sum_{\ell} e^{i\mathbf{k} \cdot \mathbf{x}} \tilde{h}_{\ell\mathbf{k}}(t) \hat{a}_{\ell\mathbf{k}} e_{\alpha\beta}^{(\ell)}(\mathbf{k}) + \text{h.c.} \right]. \end{aligned} \quad (21)$$

In order to distinguish the mode function $\tilde{h}_{\ell\mathbf{k}}(t)$ from the classical one in Eq. (12) we have put a tilde on it. The creation and annihilation operators of each polarization state follow

$$[\hat{a}_{\ell\mathbf{k}}, \hat{a}_{\ell'\mathbf{k}'}^{\dagger}] = \delta_{\ell\ell'} \delta^3(\mathbf{k} - \mathbf{k}'), \quad (22)$$

and zero otherwise. By introducing

$$\begin{aligned} \hat{h}_{\ell}(\mathbf{x}, t) &\equiv \frac{1}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \hat{C}_{\alpha\beta}^{(t)}(\mathbf{x}, t; \mathbf{k}) e^{(\ell)\alpha\beta}(\mathbf{k}) \\ &= \int \frac{d^3k}{(2\pi)^{3/2}} \left[e^{i\mathbf{k} \cdot \mathbf{x}} \tilde{h}_{\ell\mathbf{k}}(t) \hat{a}_{\ell\mathbf{k}} + \text{h.c.} \right], \end{aligned} \quad (23)$$

Eq. (3) can be written as

$$\delta^2 S_g = \int a^3 F \sum_{\ell} \left(\dot{\hat{h}}_{\ell}^2 - \frac{1}{a^2} \hat{h}_{\ell}^{\gamma} \hat{h}_{\ell, \gamma} \right) dt d^3x. \quad (24)$$

The equation of motion becomes

$$\ddot{\hat{h}}_{\ell} + \left(3H + \frac{\dot{F}}{F} \right) \dot{\hat{h}}_{\ell} - \frac{1}{a^2} \nabla^2 \hat{h}_{\ell} = 0. \quad (25)$$

The conjugate momenta are

$$\delta \hat{\pi}_{h_{\ell}}(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\hat{h}}_{\ell}} = 2a^3 F \dot{\hat{h}}_{\ell}. \quad (26)$$

From the equal time commutation relation between \hat{h}_ℓ and $\delta\hat{\pi}_{h_\ell}$ we have

$$[\hat{h}_\ell(\mathbf{x}, t), \dot{\hat{h}}_\ell(\mathbf{x}', t)] = \frac{i}{2a^3 F} \delta^3(\mathbf{x} - \mathbf{x}'). \quad (27)$$

From Eqs. (23,27) we have

$$\tilde{h}_{\ell\mathbf{k}}(t)\dot{\tilde{h}}_{\ell\mathbf{k}}^*(t) - \tilde{h}_{\ell\mathbf{k}}^*(t)\dot{\tilde{h}}_{\ell\mathbf{k}}(t) = \frac{i}{2a^3 F}. \quad (28)$$

For $n_g = \text{constant}$ Eq. (25) has an exact solution as in Eq. (8). In terms of the mode function we have

$$\tilde{h}_{\ell\mathbf{k}}(\eta) = \frac{\sqrt{\pi|\eta|}}{2a} \left[c_{\ell 1}(\mathbf{k}) H_{\nu_g}^{(1)}(k|\eta|) + c_{\ell 2}(\mathbf{k}) H_{\nu_g}^{(2)}(k|\eta|) \right] \sqrt{\frac{1}{2F}}, \quad (29)$$

where according to the normalization condition in Eq. (28) the coefficients $c_{\ell 1}(\mathbf{k})$ and $c_{\ell 2}(\mathbf{k})$ follow

$$|c_{\ell 2}(\mathbf{k})|^2 - |c_{\ell 1}(\mathbf{k})|^2 = 1. \quad (30)$$

We introduce the power spectrum of the Hilbert space gravitational wave operator based on the vacuum expectation value as

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}(\mathbf{k}, t) \equiv \frac{k^3}{2\pi^2} \int \langle \hat{C}_{\alpha\beta}^{(t)}(\mathbf{x} + \mathbf{r}, t) \hat{C}^{(t)\alpha\beta}(\mathbf{x}, t) \rangle_{\text{vac}} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r, \quad (31)$$

where $\langle \rangle_{\text{vac}} \equiv \langle \text{vac} | \text{vac} \rangle$ is a vacuum expectation value with $\hat{a}_{\ell\mathbf{k}}|\text{vac}\rangle \equiv 0$ for all \mathbf{k} . We can show

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}(\mathbf{k}, t) = 2 \sum_{\ell} \mathcal{P}_{\hat{h}_\ell}(\mathbf{k}, t) = 2 \sum_{\ell} \frac{k^3}{2\pi^2} |\tilde{h}_{\ell\mathbf{k}}(t)|^2. \quad (32)$$

In the Einstein gravity limit, where $F = 1/(8\pi G)$, each \hat{h}_ℓ in Eq. (24) can be corresponded to a minimally coupled scalar field without potential with a normalization $\hat{h}_\ell = \hat{\phi}/\sqrt{2F} = \sqrt{4\pi G}\hat{\phi}$; in this case, assuming equal contributions from each polarization, the power spectrum becomes

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2} = 2\mathcal{P}_{\hat{h}_\ell}^{1/2} = \sqrt{16\pi G}\mathcal{P}_{\hat{\phi}}^{1/2}. \quad (33)$$

Using Eq. (17) we can show that the conformal transformations of Eqs. (21-32) also reproduce the equations valid in Einstein gravity.

IV. VACUUM FLUCTUATIONS

In this section we will apply the quantum formulation of the gravitational wave in generalized gravity to various specific cosmological situations. We will derive the quantum vacuum fluctuations in several interesting cosmological models in analytic forms. This analytic treatments are possible mainly because the background models will satisfy the $n_g = \text{constant}$ condition, thus allowing exact solutions in Eq. (29). The corresponding vacuum fluctuations of the scalar type perturbation are presented in [13]. Since the equations and the solutions of the background models are presented in Sec. III of [13], in the following we summarize the background solutions without deriving them again.

A. Minimally coupled scalar field

The minimally coupled scalar field is a case of Eq. (1) with $f = R/(8\pi G)$, thus $F = 1/(8\pi G)$, and $\omega = 1$. We have $\epsilon_3 = 0$. The cases where the background scale factor follows an exponential or a power-law expansion in time correspond to $\dot{\epsilon}_1 = 0$. Thus n_g is a constant and the solution in Eq. (29) applies.

1. Exponential expansion

For $a \propto e^{Ht}$ with $H = \text{constant}$ the background has a solution with $V = \text{constant}$ and $\dot{\phi} = 0$. We have $\epsilon_1 = 0$, thus $n_g = 2$ and $\nu_g = \frac{3}{2}$. From Eqs. (29,32) we can present the power spectrum valid in general scale. In the large scale limit we have

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, \eta) = \sqrt{16\pi G} \frac{H}{2\pi} \sqrt{\frac{1}{2} \sum_{\ell} |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2}. \quad (34)$$

In Eq. (34) and in the following, $c_{\ell 1}(\mathbf{k})$ and $c_{\ell 2}(\mathbf{k})$ satisfy Eq. (30). If we have equal contributions from the two polarization states, the dependence on the vacuum state in Eq. (34) and the following can be written as

$$\sqrt{\frac{1}{2} \sum_{\ell} |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2} = |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|. \quad (35)$$

The often favored vacuum state in the literature corresponds to a special (and the simplest) case with $c_{\ell 2}(\mathbf{k}) \equiv 1$ and $c_{\ell 1}(\mathbf{k}) \equiv 0$, thus

$$\sqrt{\frac{1}{2} \sum_{\ell} |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2} = 1. \quad (36)$$

Equation (34) with the simplest vacuum choice was derived in [5–10].

2. Power-law expansion

For $a \propto t^p$ with $p = \text{constant} > 1$ the background has a solution with $\dot{\phi} = \sqrt{2p}/t$ and $V = p(3p-1)/t^2 \propto e^{-\sqrt{2/p}\phi}$, [22]. In this case we have $\epsilon_1 = -1/p$, thus $\nu_g = \frac{3p-1}{2(p-1)}$; ν_g is the same as the corresponding index appearing in the scalar type perturbation in [13]. The general power spectrum follows from Eqs. (29,32). In the large scale limit we have

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, \eta) = \sqrt{16\pi G} \frac{H}{2\pi} \frac{\Gamma(\nu_g)}{\Gamma(3/2)} \frac{p-1}{p} \left(\frac{2}{k|\eta|} \right)^{\nu_g-3/2} \sqrt{\frac{1}{2} \sum_{\ell} |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2}. \quad (37)$$

In the limit of $p \rightarrow \infty$ Eq. (37) reproduces Eq. (34). Equation (37) in the simplest vacuum choice was first derived in [6].

3. Potential-less case

In a case with vanishing potential, $V(\phi) = 0$, we have a solution with $a \propto t^{1/3}$ which corresponds to a stiffest equation of state in the case of ideal fluid with $p = \mu$. This case does not produce any inflation. However, we could calculate the exact power spectrum based on the vacuum expectation value. We have $\epsilon_1 = -3$, thus $n_g = -\frac{1}{4}$ and $\nu_g = 0$. The general power spectrum follows from Eqs. (29,32). In the large scale limit we have

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, \eta) = \sqrt{16\pi G} \left(\sqrt{\frac{4|\eta|}{a^2}} \right)_1 \left(\frac{k}{2\pi} \right)^{3/2} \ln(k|\eta|) \sqrt{\frac{1}{2} \sum_{\ell} |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2}, \quad (38)$$

where the subindex 1 indicates that since the quantity does not depend on time we have evaluated it in an arbitrary time t_1 . Later in this section we will see that the potential-less situations in several generalized gravity theories also produce the same k^3 dependence of the power spectra as in Eq. (38).

B. Scalar-tensor theory

A scalar-tensor theory is given by an action [23]

$$S = \int d^4x \sqrt{-g} \left[\phi R - \omega(\phi) \frac{\phi^{;a} \phi_{;a}}{\phi} - V(\phi) \right], \quad (39)$$

which is a case of Eq. (1) with $F = 2\phi$. Ignoring the potential term and for $\omega = \text{constant}$ we have [24]:

$$a \propto |t_0 - t|^{-q}, \quad \phi \propto |t_0 - t|^{1+3q}, \quad q \equiv -\frac{1 + \omega \mp \sqrt{1 + \frac{2}{3}\omega}}{4 + 3\omega}. \quad (40)$$

A pole-like acceleration stage can be realized when $q > 0$ which corresponds to the upper sign and $t_0 > t$. In the following analyses, for generality we will consider *both* signs. From Eq. (9) we have $\epsilon_1 = 1/q$ and $\epsilon_3 = -(1 + 3q)/(2q)$, thus $n_g = -\frac{1}{4}$ and $\nu_g = 0$. The general power spectrum follows from Eqs. (29,32). In the large scale limit we have

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, \eta) = \left(\sqrt{\frac{4|\eta|}{a^2\phi}} \right)_1 \left(\frac{k}{2\pi} \right)^{3/2} \ln(k|\eta|) \sqrt{\frac{1}{2} \sum_{\ell} |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2}. \quad (41)$$

C. Induced gravity theory

The induced gravity theory is given by [25]

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \epsilon \phi^2 R - \frac{1}{2} \phi^{;a} \phi_{;a} - V(\phi) \right], \quad (42)$$

which is a case of Eq. (1) with $F = \epsilon \phi^2$. Assuming $V = 0$ we have the following background solution [26]:

$$a \propto |t_0 - t|^{-q}, \quad \phi \propto |t_0 - t|^{\frac{1+3q}{2}}, \quad q \equiv -\frac{1 + 4\epsilon \mp 4\epsilon \sqrt{1 + \frac{1}{6\epsilon}}}{3 + 16\epsilon}. \quad (43)$$

Cases with $q > 0$, thus the upper sign, and $t_0 > t$ include the pole-like acceleration stage. For generality, we will take both signs. From Eq. (9) we have $\epsilon_1 = 1/q$ and $\epsilon_3 = -(1 + 3q)/(2q)$, thus $n_g = -\frac{1}{4}$ and $\nu_g = 0$. The general power spectrum follows from Eqs. (29,32). In the large scale limit we have

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, \eta) = \left(\sqrt{\frac{8|\eta|}{\epsilon a^2 \phi^2}} \right)_1 \left(\frac{k}{2\pi} \right)^{3/2} \ln(k|\eta|) \sqrt{\frac{1}{2} \sum_{\ell} |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2}. \quad (44)$$

D. String theory

The low-energy effective action of string theory is given by [27]

$$S = \int d^4x \sqrt{-g} \frac{1}{2} e^{-\phi} (R + \phi^{;a} \phi_{;a}), \quad (45)$$

which is a case of Eq. (1) with $F = e^{-\phi}$. For the background we have a solution

$$a \propto |t_0 - t|^{\mp 1/\sqrt{3}}, \quad e^{\phi} \propto |t_0 - t|^{-1 \mp \sqrt{3}}. \quad (46)$$

The upper sign with $t < t_0$ represents a pole-like inflation stage which is called as a pre-big bang stage in [28]. In the following we consider both signs for generality. From Eq. (9) we have $\epsilon_1 = \pm\sqrt{3}$ and $\epsilon_3 = -(3 \pm \sqrt{3})/2$, thus $n_g = -\frac{1}{4}$ and $\nu_g = 0$. The general power spectrum follows from Eqs. (29,32). In the large scale limit we have

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, \eta) = \left(\sqrt{\frac{8|\eta|}{a^2 e^{-\phi}}} \right)_1 \left(\frac{k}{2\pi} \right)^{3/2} \ln(k|\eta|) \sqrt{\frac{1}{2} \sum_{\ell} |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2}. \quad (47)$$

The gravitational wave power spectrum in the pre-big bang scenario has been studied in [18].

E. Conformal transformation

As generally have shown in Secs. II,III the conformal transformation relates the results between the generalized gravity theories and Einstein gravity. For individual generalized gravity theories in Secs. IV B-IV D, using the conformal transformation properties in Eq. (17), we can show that the conformal transformations of the results in Secs. IV B-IV D reproduce the results in Sec. IV A 3.

V. GRAVITATIONAL WAVE SPECTRUMS

From Eq. (6) we find that in the large scale limit the growing mode of $C_{\alpha\beta}^{(t)}$ (and h_ℓ) is conserved

$$h_\ell(\mathbf{x}, t) = C_\ell(\mathbf{x}). \quad (48)$$

We emphasize that Eq. (48) is valid for general changes in $V(\phi)$, $\omega(\phi)$ and $f(\phi, R)$ in generalized gravity [11], and the change in the equation of state $p = p(\mu)$ in the fluid era [16]. This reflects the kinematic nature of the evolution in the superhorizon scale. Equation (48) is valid in a scale larger than the visual horizon.

The gravitational waves in the observationally relevant scales are stretched into superhorizon scale during the inflation era. The evolution of the gravitational wave in the superhorizon scale is characterized by the above temporal conservation. Thus, the classical spectrum of the gravitational wave in later epochs can be easily derived. In the literature, the popular method is based on matching the Bogoliubov coefficients assuming the sudden jump transitions among different cosmological eras (the inflation era, the radiation era, and the matter era) [8,10]; in the case of a scalar type perturbation the sudden jump approximation using the joining variables is made in [29]. However, since the conservation variable is applicable considering general changes in the background state of the universe (the equation of state, the field potential, and even the gravity theories) we can derive the same result in a more direct manner: As the quantum fluctuation of $\hat{C}_{\alpha\beta}^{(t)}$ becomes superhorizon scale and classicalizes, it can somehow be related to the classical fluctuation of $C_{\alpha\beta}^{(t)}$, and thereafter, simply is conserved as long as the scale remains in the superhorizon scale. As the fluctuation transits back into horizon scale in the matter era in Einstein gravity, the solution in Eq. (18) will apply.

Comparing Eqs. (31,32) and Eqs. (15,16) we are tempted to take the following *ansatz*: in the large scale limit during inflation era we assume

$$\mathcal{P}_{h_\ell}(\mathbf{k}, \eta) \equiv \mathcal{P}_{\hat{h}_\ell}(\mathbf{k}, \eta) \times Q_\ell(\mathbf{k}), \quad (49)$$

where $Q_\ell(\mathbf{k})$ is a classicalization factor for the gravitational wave with a polarization state ℓ . $Q_\ell(\mathbf{k})$ may take into account of the possible nontrivial effects which arise during the classicalization processes of the fluctuating quantum gravitational wave field, [30]. If we assume equal contributions from both polarizations, we have

$$\mathcal{P}_{C_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, \eta) = \mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, \eta) \times \sqrt{Q_\ell(k)}. \quad (50)$$

In Eqs. (49,50) the right hand side should be evaluated while the scale is in the large scale limit during the inflation era. The power spectrums of the vacuum fluctuations, $\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}$, for several expansion stages are derived in Sec. IV.

The classicalized gravitational wave in the inflation era is, later on, preserved as in Eq. (48) as long as the scale remains in the superhorizon scale. As emphasized, the conserved behavior holds including the transitions between the inflation era (possibly based on generalized gravity) and the radiation era, and between the radiation era and the matter era. During the matter era we have a solution in Eq. (18). Thus, by directly comparing Eqs. (19,49) in the superhorizon scales, and using one of the power spectrums in Sec. IV as the seed generating mechanism, we can determine the classical amplitude of the gravitational wave spectrum, $A_T(\mathbf{k})$, in Eq. (20). As an example, in a scenario with exponential inflation based on Einstein gravity, from Eqs. (19,49,34) we can show that

$$A_T^{1/2}(\mathbf{k}) = \frac{\pi}{\sqrt{2}} \sqrt{16\pi G \frac{H}{2\pi}} \sqrt{\frac{1}{2} \sum_\ell |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2} \times Q_\ell(\mathbf{k}). \quad (51)$$

If we ignore the dependences on the choice of the vacuum state and the classicalization factor, we have

$$\sqrt{\frac{1}{2} \sum_\ell |c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})|^2} \times Q_\ell(\mathbf{k}) = 1. \quad (52)$$

In this case we have $A_T(\mathbf{k}) = 2\pi GH^2 \propto k^0$. Thus, from Eq. (20) we have a scale invariant spectrum with $n_T = 0$. Similarly, in the power-law inflation in Einstein gravity, using Eq. (37) we have $n_T = 3 - 2\nu_g = 2/(1 - p)$. In the potential-less case in Eq. (38) and also in the various pole-like inflation models in Eqs. (41,44,47), additionally ignoring the logarithmically dependent term, we have a common spectrum with

$$n_T = 3. \quad (53)$$

This spectrum fundamentally differs from the scale invariant one with $n_T = 0$. We, however, point out that the choice of the physically relevant vacuum state and the classicalization factor could possibly dominate both the amplitude and the spectral dependence of the generated power spectrum.

The propagation of the relic radiation will be affected by the classically fluctuating *spacetime* caused by the classical gravitational wave. In our scenario the classical wave may have the origin in the inflation era as the linear order quantum fluctuations of the spacetime are stretched to superhorizon scale and are classicalized. The effects of the curved spacetime (including effects due to the gravitational wave) on the propagation of the relic cosmic microwave background photons cause an important observational consequence. The gravitational redshifts of the relic photons cause temperature fluctuations of the present day microwave photons in different directions [2]

$$\frac{\delta T}{T}(\mathbf{e}; \mathbf{x}_R) = - \int_{\lambda_e}^{\lambda_o} \left[\frac{\partial}{\partial \eta} C_{\alpha\beta}^{(t)}(\mathbf{x}(\lambda), \eta) \right] e^\alpha e^\beta d\lambda, \quad (54)$$

where \mathbf{e} is a unit vector in the direction of the relic photon; $\lambda_e = \lambda_{\text{emitted}}$ and $\lambda_o = \lambda_{\text{observed}}$. We have parametrized the path of the photon so that $\mathbf{x}(\lambda) = x(\lambda)\mathbf{e} - \mathbf{x}_R$, where $x(\lambda) = \eta_o - \lambda$ and \mathbf{x}_R is the location of the observer. We expand

$$\frac{\delta T}{T}(\mathbf{e}; \mathbf{x}_R) = \sum_{lm} a_{lm}(\mathbf{x}_R) Y_{lm}(\mathbf{e}). \quad (55)$$

We can derive the rotationally symmetric quantity $\langle a_l^2 \rangle \equiv \langle |a_{lm}(\mathbf{x}_R)|^2 \rangle_{\mathbf{x}_R}$, where $\langle \rangle_{\mathbf{x}_R}$ is an average over every possible location of the observer. Using Eqs. (55,54,12), and after a little but well known algebra (for example, see [10]), we can derive [6,7]

$$\begin{aligned} \langle a_l^2 \rangle &= \frac{2}{\pi} \frac{\Gamma(l+3)}{\Gamma(l-1)} \int_0^\infty \left| \int_{\eta_e}^{\eta_o} \frac{\partial h_{\mathbf{k}}(\eta)}{\partial \eta} \frac{j_l(k\eta_o - k\eta)}{(k\eta_o - k\eta)^2} k d\eta \right|^2 dk \\ &= \frac{9\pi}{2} \frac{\Gamma(l+3)}{\Gamma(l-1)} \int_0^\infty A_T(k) |I_l(k)|^2 d \ln k, \\ I_l(k) &\equiv \frac{2}{\pi} \int_{\eta_e}^{\eta_o} \frac{j_2(k\eta)}{k\eta} \frac{j_l(k\eta_o - k\eta)}{(k\eta_o - k\eta)^2} k d\eta, \end{aligned} \quad (56)$$

where in the second step we used Eq. (18). In Eq. (56) we have assumed equal contributions from both polarizations. In combination with the derived $A_T(\mathbf{k})$ for each seed generating mechanism in Sec. IV, for example Eq. (51) in the exponential inflation scenario, the multipole moments of the temperature anisotropy can be known by evaluating Eq. (56).

VI. DISCUSSIONS

In this paper we have considered the inflation models based on generalized gravity theories. We have derived the seed spectrums of the gravitational wave based on the vacuum expectations of the quantized fluctuating metric in analytic forms. In generalized gravity theories the general background evolutions under the potential-less assumption lead to common spectrums for the gravitational wave and for the scalar type perturbation as

$$n_T \simeq 3, \quad n \simeq 4, \quad (57)$$

whereas the observationally favored [31] scale-invariant spectrums based on the near exponential inflation are $n_T \simeq 0$ and $n \simeq 1$; n is a spectral index for the density perturbation defined in [13].

The proper [which means the mathematical convenience at least] handling of the scalar type perturbation is possible by a following gauge invariant combination of variables

$$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi \equiv -\frac{\dot{\phi}}{H}\varphi_{\delta\phi}, \quad (58)$$

where $\delta\phi(\mathbf{x}, t)$ is the perturbed part of the scalar (or dilaton) field, and $\varphi(\mathbf{x}, t)$ is the perturbed part of the spatial scalar curvature, see Eqs. (3,4) of [16]. Thus the gauge invariant combinations $\delta\phi_\varphi$ and $\varphi_{\delta\phi}$ are the same as $\delta\phi$ in the uniform-curvature gauge and φ in the uniform-field gauge, respectively; the uniform-field gauge coincides with the comoving gauge in the limit of the minimally coupled scalar field. As thoroughly presented in [11–13] the scalar type perturbation in the large scale limit is also characterized by a conserved quantity, which is $\varphi_{\delta\phi}$. We also have general asymptotic solutions and the exact analytic solution under a certain, but generally applicable, condition. By comparing the power spectrums in Sec. IV with the ones of the scalar type perturbation in Sec. III of [13] we can show the following strikingly similar structures. In the three cases of the minimally coupled scalar field in Sec. IV A, by taking the simplest vacuum choice, we have

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, t) = \sqrt{16\pi G} \times \mathcal{P}_{\delta\hat{\phi}_\varphi}^{1/2}(\mathbf{k}, t). \quad (59)$$

Similarly, in the three cases of the generalized gravity theories in Secs. IV B–IV D and the case in Sec. IV A 3, also by taking the simplest vacuum choices, we have

$$\mathcal{P}_{\hat{C}_{\alpha\beta}^{(t)}}^{1/2}(\mathbf{k}, t) = 2\sqrt{3} \times \mathcal{P}_{\hat{\phi}_{\delta\phi}}^{1/2}(\mathbf{k}, t). \quad (60)$$

Recently, there have been many attempts to reconstruct the inflationary potential from the observed large scale structure and the microwave anisotropy, [32]. These attempts aim a high accuracy reconstruction of the inflaton potential. In such cases one may be not allowed to ignore two potentially important effects which have been always ignored in the literature: the spectral dependences on the vacuum choice and on the classicalization factor. These two effects could separately occur for both the gravitational wave and the scalar type perturbation [13], and could possibly dominate the spectrums. The accurate determination of the classicalization factors may require the treatment which goes beyond the linear approximation of the quantum fluctuations. The vacuum choices in the curved spacetime may also depend on concrete physical arguments.

We would like to emphasize that the formulation made in Secs. II, III in a unified way is generally applicable to a broad class of gravity theories included in Eq. (1).

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